TRINITY COLLEGE

Year 12 Mathematics Specialist 3,4
Test 2 2021

Section 1 Calculator Free

Sketching Rational Graphs and Vectors in 3D Introduction

STUDENT'S NAME Solutions

DATE: Monday 29 March **TIME:** 19 minutes **MARKS**: 19

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (3 marks)

Determine the angle between the two planes $\underset{\sim}{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} = 4$ and $\underset{\sim}{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 7$. You may express your answer in terms of an inverse trigonometric function.

Angle between the two normals

$$\Rightarrow \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \sqrt{10}\sqrt{9}\cos\theta$$

✓ dot product
✓ magnitudes
✓ inverse cos.

$$\Rightarrow \frac{4}{3\sqrt{10}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{4}{3\sqrt{10}}\right)$$

2. (4 marks)

Determine the domain and range for $f(x) = \dfrac{1}{\sqrt{x-1}-1}$

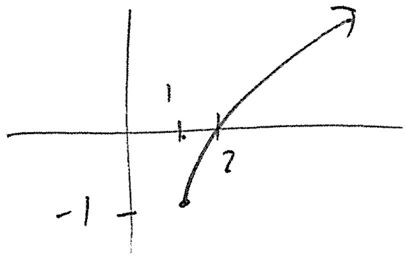
let $g(x) = \sqrt{x-1} - 1$

✓ graph $g(x)$

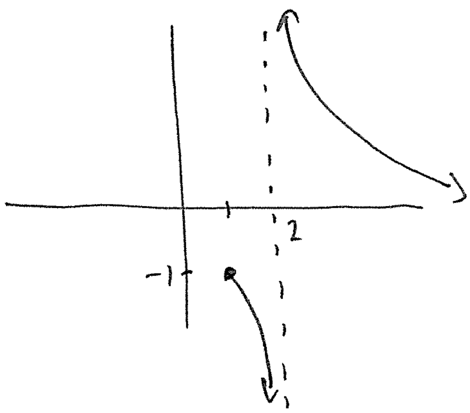
✓ graph of $f(x)$

✓ domain

✓ range



$\therefore \dfrac{1}{\sqrt{x-1}-1}$



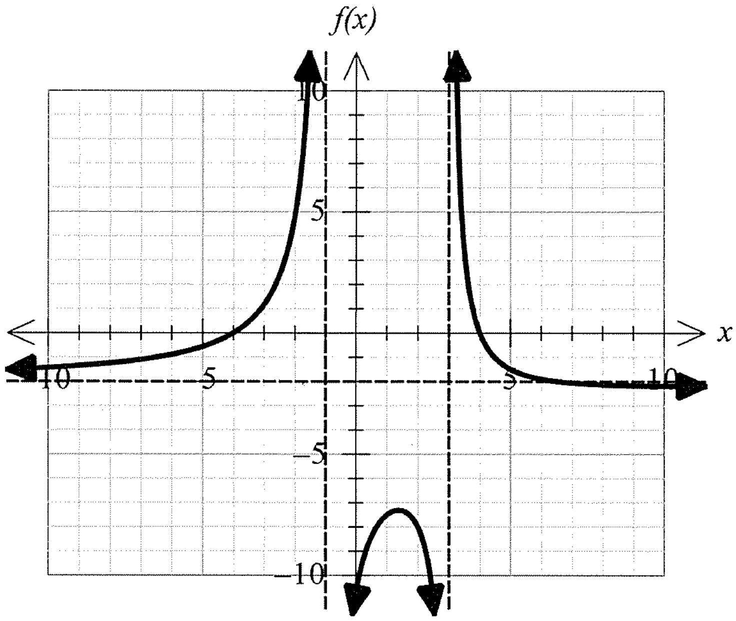
$D: \{x : x \in \mathbb{R},\ x \geq 1,\ x \neq 2\}$

$R: \{y : y \in \mathbb{R},\ y \leq -1,\ y > 0\}$

3. (6 marks)

The graph of $y = f(x)$ is show on the axes below. The defining rule is given by

$f(x) = \dfrac{-a(x^2 - b)}{(x+c)(x-d)}$ where a, b, c and d are positive constants.



Determine the value of the constants a, b, c and d. Justify your answers.

a	b	c	d
2	16	1	3

Horizontal asymptote $y = -2$ $\Rightarrow$ $a = 2$ ✓ justify ✓ answer

x-int is $(4, 0)$ $\Rightarrow$ $0 = -2(4^2 - b)$ ✓ justify

$\Rightarrow$ $b = 16$ ✓ answer

Vertical asymptotes at $x = -1$ and $x = 3$

$\Rightarrow$ $(x+1)$ and $(x-3)$ ✓ justify

✓ answer

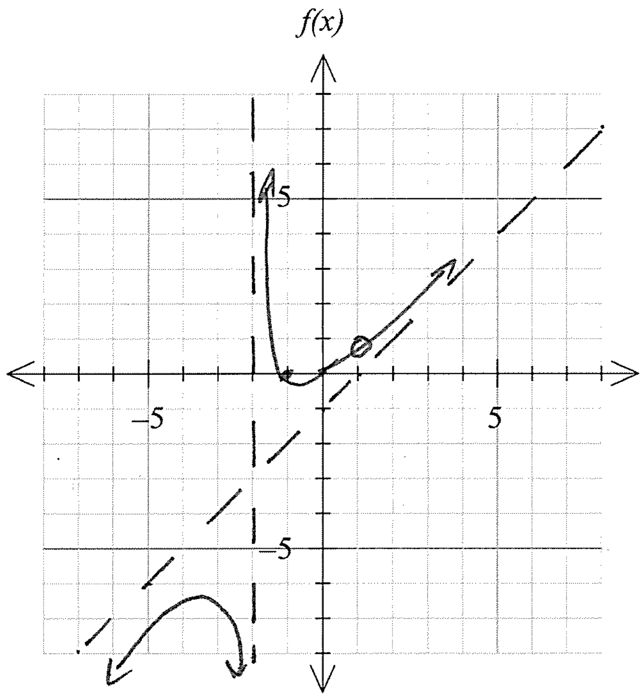
$\Rightarrow$ $c = 1$ $d = 3$

4. (6 marks)

Sketch the function $f(x)=\dfrac{x^3-x}{(x+2)(x-1)}$, showing all intercepts, holes, poles and asymptotes.

It is not necessary to identify any stationary points.

✓ factorise



✓ algebraic long division

✓ oblique

✓ asymptote $x=-2$

✓ intercepts & shape

✓ hole

$$f(x) = \frac{x(x+1)\cancel{(x-1)}}{(x+2)\cancel{(x-1)}}, \quad x \neq 1$$

$$= \frac{x^2+x}{x+2}$$

y-int $\Rightarrow$ $(0,0)$

x-int $\Rightarrow$ $0 = x(x+1)$

$\Rightarrow$ $x = 0, -1$

Now

$$\begin{array}{r} x \quad -1 \\ x+2 \,\big)\, \overline{x^2 + x + 0} \\ \underline{x^2 \quad 2x + 0} \\ -x \\ \underline{-x - 2} \\ 2 \end{array}$$

$$\therefore \quad f(x) = \frac{2}{x+2} + x - 1$$

TRINITY COLLEGE

Year 12 Mathematics Specialist 3,4
Test 2 2021

Section 2 Calculator Assumed
Sketching Rational Graphs and Vectors in 3D Introduction

STUDENT'S NAME ______________________________

DATE: Monday 29 March **TIME:** 31 minutes **MARKS**: 31

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (3 marks)

(a) Plane Π has the equation $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$ and a sphere has the vector equation $|\underset{\sim}{r} - \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}| = k$

Describe geometrically what happens for different values of k. [2]

k varies the radius of the sphere. ✓
For different values of k it will either
intersect the plane, be a tangent, or
be above or below the plane. ✓

(b) If the cross product of two vectors is $\underset{\sim}{0}$, describe the geometric relationship between the two vectors. [1]

The two vectors are parallel. ✓

6. (10 marks)

A plane Π contains the three points $(1,2,3)$, $(4,5,6)$ and $(-2,3,1)$

(a) Determine a normal to the plane Π [3]

dir $\quad \underset{\sim}{b} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✓ dir

dir $\quad \underset{\sim}{c} = \begin{pmatrix} -6 \\ -2 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$ ✓ dir

$\underset{\sim}{n} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$ ✓ cross product

(b) Determine the ~~vector~~ equation of the plane Π in Cartesian form [3]

$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$

$\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$ ✓ scalar eqn

$\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = -7$ ✓ vector eqn

Cartesian $\quad 3x + y - 4z = -7$ ✓ Cartesian eqn

A line, L_1, has Cartesian equation $x-2=y+3=\frac{z-1}{2}$.

(c) Determine the vector equation of the line in the form $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{b}$ [2]

$$\Rightarrow \quad \lambda = x-2 \qquad \Rightarrow \quad \lambda+2 = x$$
$$\lambda = y+3 \qquad \lambda-3 = y$$
$$\lambda = \frac{z-1}{2} \qquad 2\lambda+1 = z$$

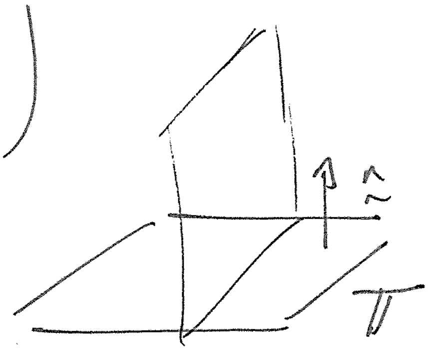
✓ parametric

$$\therefore \quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

✓ vector

(d) Determine the equation of the plane that is perpendicular to plane Π and contains line L_1 [2]

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$



✓ line

✓ uses normal for second direction

OR

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -6 \\ 10 \\ -4 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} 6 \\ -10 \\ 4 \end{pmatrix} = 44$$

7. (10 marks)

A sphere has equation $x^2 + y^2 + z^2 - 2x + 4z = 0$ and a line has equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$

(a) Determine the vector equation of the sphere. [3]

$$x^2 - 2x + y^2 + z^2 + 4z = 0$$

$$(x-1)^2 - 1 + y^2 + (z+2)^2 - 4 = 0$$

$$(x-1)^2 + y^2 + (z+2)^2 = 5$$

✓ completes square

✓ centre

✓ radius

Vector eqn $\left| \underset{\sim}{r} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right| = \sqrt{5}$

(b) Determine the point(s) of intersection of the line and the sphere. [4]

$$\Rightarrow \left| \begin{pmatrix} 1+\lambda \\ 2-2\lambda \\ 3-3\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right| = \sqrt{5}$$

✓ substitute

✓ λ

Solving $\lambda = 1, \frac{12}{7}$

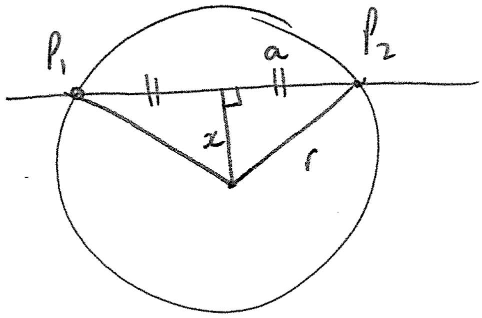
✓ P_1

✓ P_2

$\therefore$ $P_1 \big|_{\lambda=1} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

$P_2 \big|_{\lambda=\frac{12}{7}} = \frac{1}{7}\begin{pmatrix} 19 \\ -10 \\ -15 \end{pmatrix}$ or $\begin{pmatrix} 2.71 \\ -1.43 \\ -2.14 \end{pmatrix}$

(c) Determine the closest distance between the line and the centre of the sphere. [3]



$$|P_1 - P_2| = \left| \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2.71 \\ -1.43 \\ -2.14 \end{pmatrix} \right|$$

$$= \frac{5\sqrt{14}}{7} \quad \text{or} \quad 2.67$$

✓ dist $|P_1 - P_2|$

$$\therefore \quad a = \frac{5\sqrt{14}}{14}$$

Now by pythagoras

✓ halves & uses pythagoras

$$x^2 = r^2 - a^2$$

$$= (\sqrt{5})^2 - \left(\frac{5\sqrt{14}}{14}\right)^2$$

✓ answ

$$x^2 = \frac{45}{14}$$

$$\therefore \quad x = \frac{3\sqrt{70}}{14} \quad \text{or} \quad 1.79$$

8. ($\cancel{8}$ 9 marks)

Plane Π has Cartesian equation $y = 8x - 4z + 9$.

(a) Determine a vector normal to the plane Π. [2]

$$8x - y - 4z = -9$$ ✓ rearrage

$$\therefore \quad \underset{\sim}{n} = \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix}$$ ✓ normal

A sphere of radius 9 is tangential to the plane Π. The point $(-2,2,9)$ lies on the surface of the sphere. The centre of the sphere has coordinates $(-9,-2,k)$, where $k < 10$.

(a) Determine the value of k [$\cancel{2}$] 3

Vect eqn sphere

$$\left| \underset{\sim}{r} - \begin{pmatrix} -9 \\ -2 \\ k \end{pmatrix} \right| = 9$$ ✓ vect eqn

Sub pt into eqn

$$\left| \begin{pmatrix} -2 \\ 2 \\ 9 \end{pmatrix} - \begin{pmatrix} -9 \\ -2 \\ k \end{pmatrix} \right| = 9$$ ✓ substitute

Solving

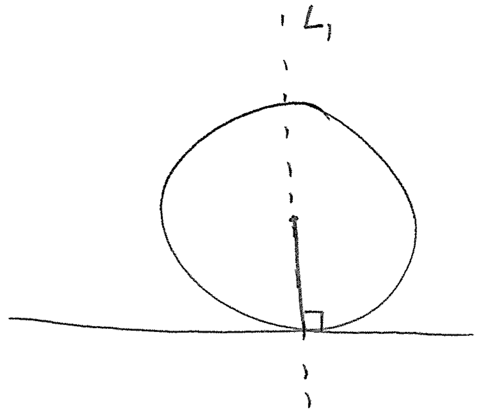
$$k = 5, 13$$ ✓ answer

$$\therefore k = 5$$

(c) Determine the coordinates of the point of intersection of the plane Π and the sphere. [4]

Plane $\underset{\sim}{r} \cdot \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix} = -9$

Sphere $\left| \underset{\sim}{r} - \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix} \right| = 9$



Eqn for L_1

$\Rightarrow \underset{\sim}{r} = \begin{pmatrix} -9 \\ -2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix}$ ✓ line

This line must intersect with the plane

$\Rightarrow \begin{pmatrix} -9+8\lambda \\ -2-\lambda \\ 5-4\lambda \end{pmatrix} \cdot \begin{pmatrix} 8 \\ -1 \\ -4 \end{pmatrix} = -9$ ✓ substitute

Solving $\lambda = 1$ ✓ λ

$\therefore$ pt is $\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ ✓ point

$\therefore$ Coordinates $(-1, -3, 1)$